[8] M. V. Schneider, "Microstrip line for microwave integrated circuits," *Bell Syst. Tech. J.*, pp. 1421–1445, May–June 1969.
[9] V. F. Fusco, *Microwave Circuit Analysis and Computer Aided Design*. Englewood Cliffs, NJ: Prentice-Hall, 1987.

A Comparison of Two Approximations for the Capacitance of a Circle Concentric with a Cross

Henry J. Riblet

Abstract —**The maximum and minimum capacitances on circles concentric with an internal cross are determined for a four-lobed as well as an eight-lobed equipotential distribution. The average and geometric mean of these extreme capacitances are then compared with the exact capacitance. The increased accuracy obtained from the eight-lobed equipotential distribution is presented in graphical form.**

I. Introduction

Oberhettinger and Magnus [1, p. 44] showed how the interior of a regular polygon can be mapped into the interior of a circle, and later Laura and Luisoni [2] showed how the exterior of a regular polygon can be mapped onto the exterior of a circle. In the latter case, circles exterior to the circle can be mapped back onto equilobed curves exterior to the polygon. Laura and Luisoni obtained power series for these transformations and showed that these equilobed curves approach circles as they become increasingly distant from the inner polygon. In this short paper, for the special case considered, it will be shown how equilobed curves of this type can be replaced by circles with only a small change in capacitance.

The case in which the inner polygon is a cross is of interest in this connection. In the first place, the exact solution to the problem of determining the capacitance of a cross in a circle is known [3, p. 1821] so that the accuracy of any approximate solution is available. Moreover, it will be shown that the method used in [1] and [2], when applied to mapping the exterior of a circle onto the exterior of cross, results in an integral which can be evaluated in closed form. This makes it a simple matter to determine the four-lobed curves bounding the region about the cross into which the area between the concentric circles is mapped. Another approximate solution to this problem is provided by the known mapping of the region between a cross and a square onto a rectangle [4]. In this case the equilobed curve surrounding the cross is eight-lobed. This short paper compares the accuracy of the four-lobed approximation with that of the eight-lobed approximation. In the first place, it is found that the relative difference between the maximum and minimum "effective" capacitances of the eight-lobed curves is about an order of magnitude lower than the same difference in the four-lobed curves, for comparable geometries. Moreover, it is shown that the average and the geometric means of the maximum and minimum "effective" capacitances on the multilobed curves are excellent approximations to the exact values. In fact, the error in the average of the maximum and minimum "effective" capacitances decreases exponentially as their relative difference decreases so that the average value for the eight-lobed case is a better approximation than the average for the four-lobed case by at least an order of magnitude, in the cases of most interest.

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II. The Exact Solution

Fig. 1 shows the successive mapping of a quadrant of a circle onto the upper half plane. The transformation

$$t = -\frac{1}{2}\left(z + \frac{1}{z}\right) \tag{1}$$

maps the upper right-hand quadrant of the circle onto the upper half plane so that two arms of the cross fall on the real and imaginary axes as shown in Fig. 1. Then the further transformation

$$w = t^2 \tag{2}$$

maps the upper right-hand quadrant of the t plane onto the upper half of the w plane. The capacitance, C, in the upper half w plane, between the line segment, fa, and the infinite line segment, bg, is given by the well-known formula [5, p. 58]

$$C = \frac{K(k)}{K'(k)} \tag{3}$$

where, in our case,

$$k^2 = \frac{(a-f)(g-b)}{(g-a)(b-f)} \tag{4}$$

$$= \frac{8\delta^2(1+\delta^4)}{(1+\delta^2)^4}. \tag{5}$$

The total capacitance, C_0, of the cross in the circle is then given by

$$C_0 = 4\frac{K(k)}{K'(k)} \tag{6}$$

where k is given by (5) and δ is the ratio of the length of the arms of the cross to the radius of the circle.

The AGM series [6, p. 331], where the convergence is very rapid, was used to calculate K and K', given k. It is values obtained in this way that are considered in this paper to be exact.

III. The Four-Lobed Case

The problem of mapping the exterior of a circle onto the exterior of a cross is solved by combining the ideas of Laura and Luisoni [2] with those of Oberhettinger and Magnus [1, p. 42]. It is found that the required transformation is given by the integral

$$w = \frac{1}{2}\int_0^Z \frac{z^4 - 1}{z^2\sqrt{z^4+1}}\,dz. \tag{7}$$

In turn, this can be integrated in closed form to give

$$w = \frac{\sqrt{z^4+1}}{2z}. \tag{8}$$

That this function maps the unit circle in the z plane of Fig. 2 onto the unit cross in the w plane is readily seen by replacing z

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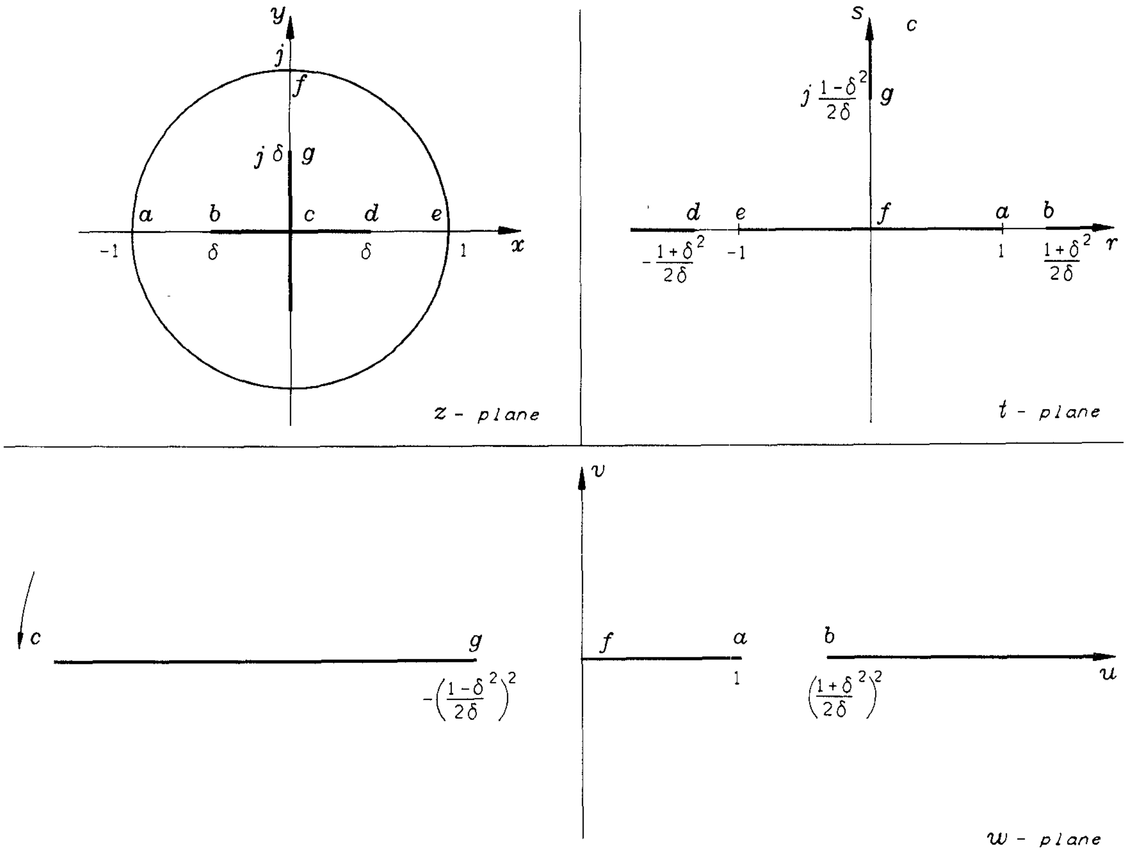

Fig. 1. z, t, and w coordinate planes.

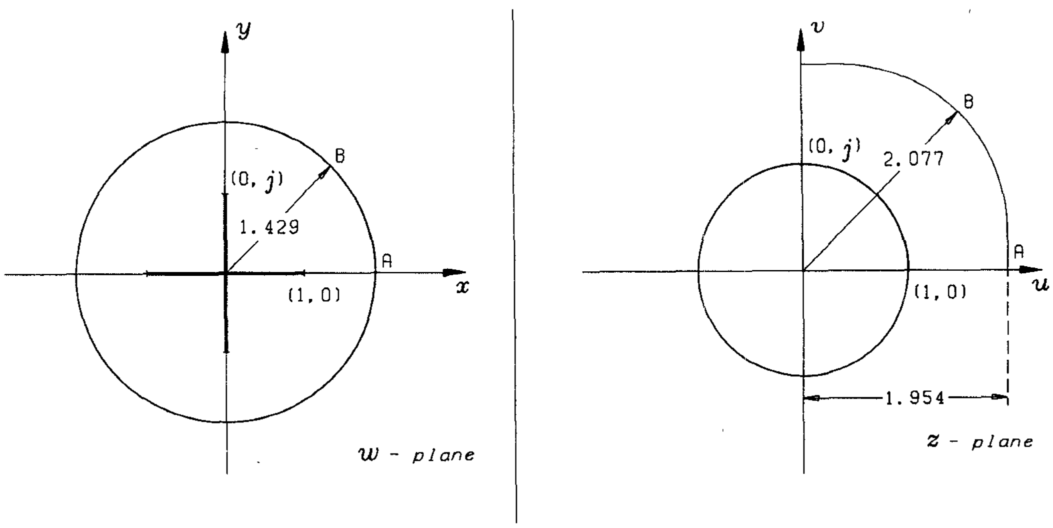

Fig. 2. w and z coordinate planes.

in (8) by $\exp(j\theta)$. Then

$$w=\sqrt{\cos 2\theta}\,. \tag{9}$$

Now (8) can readily be inverted to give

$$z=\sqrt{w^2+\sqrt{w^4-1}}\,. \tag{10}$$

Here the sign ambiguities have been resolved so that the real axis to the right of $(1,0)$ in the w plane maps into the real axis to the right of $(1,0)$ in the z plane.

Outside of the cross in the z plane of Fig. 2 is a circle, of radius 1.429, whose capacitance with respect to the cross we can approximate with the help of (10). This is done by using (10) to map circles in the w plane onto four-lobed curves in the z plane where the potential at any point with respect to the unit circle is simply the logarithm of its distance to the origin. In the example of Fig. 2, points A and B in the z plane map into points A and B in the w plane. This then determines the potential, P, of every point on the circle in the w plane with respect to the cross, since potentials are invariant under conformal transfor-

TABLE I
COMPARISON OF ACCURACY OF MEAN AND AVERAGE CAPACITANCES

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
$C_{max} - C_{min}$	0.22×10^{-4}	0.66×10^{-3}	0.53×10^{-2}	0.25×10^{-1}	0.91×10^{-1}	0.28	0.78	2.14	6.59	14.13
$(C_{max} - C_{min})/C_0$	0.94×10^{-5}	0.20×10^{-3}	0.13×10^{-2}	0.51×10^{-2}	0.15×10^{-1}	0.38×10^{-1}	0.87×10^{-1}	0.19	0.44	0.76
$C_{aver} - C_0$	$< 10^{-8}$	-0.32×10^{-7}	-0.95×10^{-6}	-0.84×10^{-5}	-0.14×10^{-4}	0.37×10^{-3}	0.50×10^{-2}	0.43×10^{-1}	0.39	1.55
$C_{mean} - C_0$	$< 10^{-8}$	-0.49×10^{-7}	-0.18×10^{-5}	-0.24×10^{-4}	-0.18×10^{-3}	-0.95×10^{-3}	-0.35×10^{-2}	-0.73×10^{-2}	0.35×10^{-1}	0.28

mation. The relationship between the geometrical capacitance, C, and the potential P, $C = 2\pi/P$, permits the association of an "effective" capacitance with every point on the circle exterior to the cross. It is clear that the maximum value of this varying capacitance will occur on the coordinate axes while the minimum value will occur just half way between them. For any value of δ, the ratio of the length of the arms of the cross to the radius of the circle (in Fig. 2, $\delta = 0.7$), it is an easy calculation to determine the maximum value of the "effective" capacitance, C_{max}, and the minimum value of the "effective" capacitance, C_{min}. Table I gives the difference between the average as well as the mean of these maximum and minimum values and the "exact" value, C_0. Not only are the average and the mean good approximations to the exact value, but the errors decrease rapidly as the relative difference, $(C_{max} - C_{min})/C_0$, between maximum and minimum values decreases. For the example considered, where $\delta = 0.7$, the relative difference between the maximum and minimum values is 0.086 while the error in the average is 0.005; but for $\delta = 0.6$, where the relative difference is 0.034, the error in the average has decreased to 0.00037. It will be shown that the error decreases exponentially with the relative difference between the maximum and minimum "effective" capacitances.

IV. THE EIGHT-LOBED CASE

Bowman [5, p. 67] has discussed the problem of mapping two noncontiguous segments of a rectangle onto two opposing sides of a rectangle. It is this method which Riblet [4] has used to map the upper right-hand square of the z plane in Fig. 2 into the rectangle in the w plane.

In the first place, the transformation

$$s = \operatorname{sn}\left(z, \sqrt{1/2}\right) \tag{11}$$

maps the upper right-hand square of the z plane into the upper right-hand quadrant of the s plane. There corresponding points are denoted by the same letters. Not shown, however, is a necessary scaling transformation which maps the square of side s into one of side $K(\sqrt{1/2})$. Then the transformation

$$t = s^2 \tag{12}$$

maps the upper right-hand quadrant of the s plane into the upper half t plane. The linear transformation

$$w = \frac{t - e}{a - e} \tag{13}$$

$$= \frac{\operatorname{cn}^2\left(D, \sqrt{1/2}\right)t + \operatorname{sn}^2\left(D, \sqrt{1/2}\right)}{\operatorname{sn}^2\left(D, \sqrt{1/2}\right)\left(1 + \operatorname{cn}^2\left(D, \sqrt{1/2}\right)\right)} \tag{14}$$

where $D = tK(\sqrt{1/2})/s$, then maps the upper half t plane into the upper half w plane as shown in Fig. 3. The transformation $v = u$ maps the upper half u plane into the upper right-hand quadrant of the v plane, which is then mapped into the *eabd* rectangle of the w plane by the elliptic integral

$$w = \int_0^v \frac{dv}{\sqrt{(1 - v^2)(1 - k^2 v^2)}} \tag{15}$$

where

$$k^2 = \operatorname{sn}^2\left(D, \sqrt{1/2}\right)\left(1 + \operatorname{cn}^2\left(D, \sqrt{1/2}\right)\right). \tag{16}$$

Given t/s, the principal computational problem of this case is the determination of the radius, R_{eq}, of that circle, as shown in the z plane of Fig. 3, which maps into an equiripple curve in the w plane. This, of course, requires the evaluation of the elliptic integral of the first kind given in (15). For this purpose the Gaussian descending transformation has been used, in which

$$k_1 = \frac{1 - k_0'}{1 + k_0'} \tag{17}$$

and

$$v_1 = \frac{1 - \sqrt{1 - k_0^2 v_0^2}}{(1 - k_0')v_0}. \tag{18}$$

Once the equiripple radius, R_{eq}, has been found by a computerized search, it only remains to search the curve in the w plane, which corresponds to the equiripple circle in the z plane, for its maximum value. The minimum values occur, by construction, at 0, 45°, and 90°.

The maximum and minimum values of the "effective" capacitance associated with the circle of equiripple radius in the z plane of Fig. 3 were obtained by dividing the maximum and minimum values on the corresponding curve of the w plane into the width of the rectangle shown in the w plane. This width is, of course, $K(k)$, where k is given by (16). These quotients were then multiplied by 4 to allow for the fact that only one quadrant of the circle in the z plane was mapped into the w plane.

Table II gives the relative difference, $(C_{max} - C_{min})/C_0$, between the maximum "effective" capacitance and the minimum "effective" capacitance as well as the difference between the average and mean "effective" capacitances and the exact capacitance, C_0, as a function of δ. When $\delta = 0.7$, the relative error is 0.0063. This is an improvement over the four-lobed case of more than an order of magnitude. The error in the average "effective" capacitance, however, is 0.000049, which is two orders of magnitude less than the similar error for the four-lobed case. The improvement is even greater for the case $\delta = 0.6$. For $\delta = 0.5$, the improvement is three orders of magnitude.

This is shown graphically in Fig. 4, where the ordinate is $\log((C_{max} - C_{min}/C_0)$ and the abscissa is $\log(C_{aver} - C_0)$. Here $C_{aver} = (C_{max} + C_{min})/2$. Here it is clear that the slope of the line joining points on the four-lobed and eight-lobed lines having the same value of δ increases as δ decreases. This figure shows that, for a given value of $(C_{max} - C_{min})/C_0$, the four-lobed

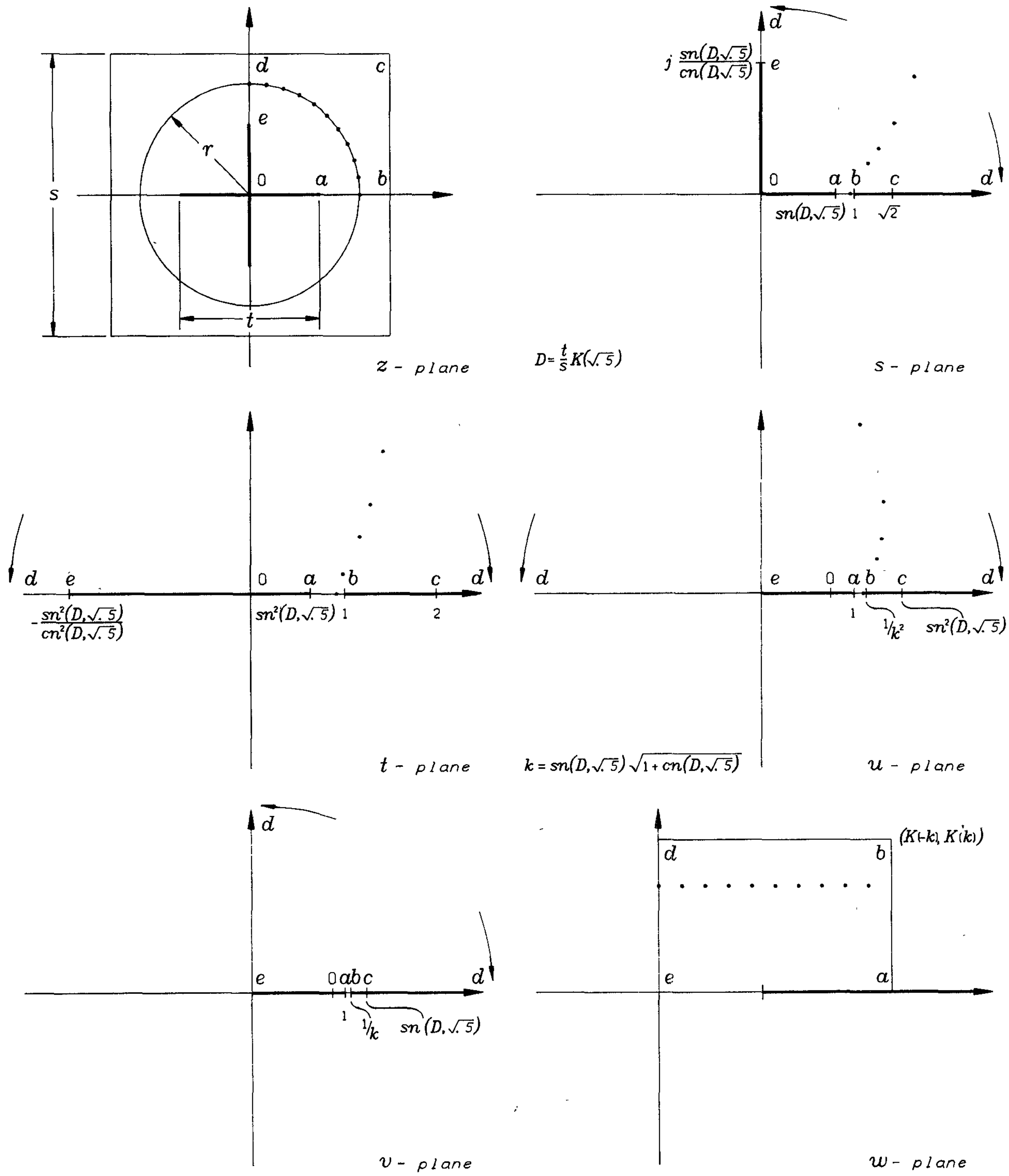

Fig. 3. z, s, t, u, v, and w coordinate planes.

TABLE II
COMPARISON OF ACCURACY OF MEAN AND AVERAGE CAPACITANCES

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
$C_{max} - C_{min}$	$< 10^{-8}$	0.35×10^{-6}	0.14×10^{-4}	0.22×10^{-3}	0.19×10^{-2}	0.12×10^{-1}	0.57×10^{-1}	0.23	0.84	1.72
$(C_{max} - C_{min})/C_0$	$< 10^{-8}$	0.11×10^{-6}	0.35×10^{-5}	0.43×10^{-4}	0.31×10^{-3}	0.16×10^{-2}	0.63×10^{-2}	0.20×10^{-1}	0.56×10^{-1}	0.92×10^{-1}
$C_{aver} - C_0$	$< 10^{-8}$	$< 10^{-8}$	$< 10^{-8}$	$< 10^{-8}$	0.15×10^{-7}	0.14×10^{-5}	0.49×10^{-4}	0.10×10^{-2}	0.16×10^{-1}	0.72×10^{-1}
$C_{mean} - C_0$	$< 10^{-8}$	$< 10^{-8}$	$< 10^{-8}$	$< 10^{-8}$	-0.58×10^{-7}	-0.92×10^{-6}	0.36×10^{-5}	0.43×10^{-3}	0.10×10^{-1}	0.52×10^{-1}

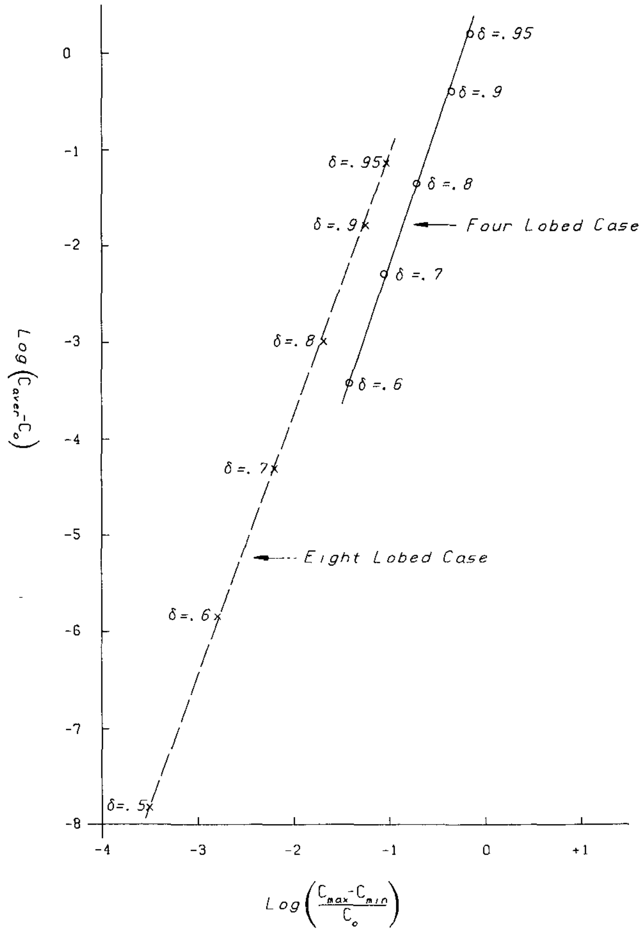

Fig. 4. Comparison of approximation errors.

approximation is about an order of magnitude more accurate than the eight-lobed approximation, but this result is offset by the reduction in the values of the $(C_{max} - C_{min})/C_0$ as δ becomes smaller.

V. Comments

It is interesting that both Table I and Table II show that $C_{mean} - C_0$ becomes positive for sufficiently large values of δ. The fact that the calculation for the four-lobed case involves, at most, the determination of K and K' using the rapidly converging AGM series virtually rules out the possibility of a numerical error. The computation for the eight-lobed case is more involved, but here great care was taken to check the accuracy of each step in the calculations. For example, each complex value of w obtained by applying the Gauss descending transformation to the given v and k was checked for accuracy by resubstituting into $v = \mathrm{sn}(w, k)$. It is believed that these results are accurate to the order of 10^{-9}.

In both cases, it will be observed that the geometric mean of C_{max} and C_{min} is a more accurate approximation than their average. It was only because the error in the average of C_{max} and C_{min} changes sign for smaller values of δ, however, that the logarithm of the error in the average is plotted in Fig. 4. Needless to say, the linearity of the data for the two cases and their parallelism were a surprise.

References

[1] F. Oberhettinger and W. Magnus, *Anwendung der Elliptischen Funktionen in Physik and Technik*. Berlin, Germany: Springer, 1949.
[2] P. A. A. Laura and L. E. Luisoni, "Approximate determination of the characteristic impedance of the coaxial system consisting of a regular polygon concentric with a circle," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 160–161, Feb. 1977.
[3] H. J. Riblet, "Expansions for the capacitance of a cross concentric with a circle with an application," *IEEE Trans. Microwave Theory Tech.*, vol. 37, Nov. 1989.
[4] H. J. Riblet, "Expansions for the capacitance of a cross concentric with a square with an identity," *IEEE Trans. Microwave Theory Tech.*, vol. 38, Oct. 1990.
[5] F. Bowman, *Introduction to Elliptic Functions with Applications*. New York: Dover Publications, 1961.
[6] Arthur Cayley, *Elliptic Functions*. New York: Dover Publications, 1961.

A New Procedure for Interfacing the Transmission Line Matrix (TLM) Method with Frequency-Domain Solutions

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***Abstract*—This paper presents a new procedure that interfaces the transmission-line matrix method (TLM) with frequency-domain solutions of electromagnetic fields. Frequency-domain solutions are transformed into appropriate time-domain sequences using the discrete Fourier transform (DFT). Hence, the corresponding boundary Johns matrix can be determined with minimum computational effort. The subsequent treatment consists in convolving the streams of TLM impulses incident on the boundary with a Johns matrix generated with the new approach. The method is applied to obtain the time-domain reflection sequence of wide-band absorbing terminations in a rectangular waveguide in the dominant mode operation. In addition, the time-domain analysis of pulse penetration through a sheet with high, but finite, conductivity is presented. Good results demonstrate the efficiency of the proposed procedure.**

I. Introduction

The transmission line matrix (TLM) method has been extensively applied to solve electromagnetic wave propagation, diffusion, and network problems in the time domain [1]–[3]. With its flexibility and the simplicity of the basic algorithm, the TLM method can handle arbitrary geometries and account for realistic features that are often neglected with other methods. Recently, two- and three-dimensional transmission line matrix microwave field simulators using new concepts and procedures have been presented [4].

In order to characterize structures with large dimensions, the TLM technique requires large memory space and CPU time. More recently, a general partitioning technique based on the Johns matrix concept [5], [6] has been developed to overcome this problem for certain applications.

In the following, a new procedure for interfacing TLM techniques with frequency-domain solutions is described: either scat-

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